# The various forms of skew-elliptical distributions and their role in finance

Adelchi Azzalini
Università di Padova, Italia

work in progress with Chris Adcock and Mauro Bernardi

Aegina workshop on
'Recent Developments in Dependence Modelling'
13–14 September 2018

# Overview

Themes:

- Symmetry modulated probability distributions
- Skew-elliptically contoured (SEC) distributions
- The many forms of SEC's, attempting clarification
- Comparison with other formulations, notably copulae

Focus is on

- *multivariate* distributions throughout
- properties and foundations rather than empirical work

# Symmetry modulation

# Symmetry modulation of distributions, general aspects

- a tool to generate probability distributions
- works by modulating/perturbing a symmetric (continuous) *baseline* distribution
- more naturally suitable for parametric formulations (although semi-parametric constructions are possible)
- here we focus on the multivariate setting
- AKA 'skew-symmetric distributions'
- Result: all distributions can be expressed in this form

# Symmetry modulation, more specifically

- Ingredients:
  - $f_0(x)$: a $d$-dimensional density, such that $f_0(x) = f_0(-x)$
  - $G_0(x)$: the CDF of a continuous univariate random variable having density symmetric about 0
  - $w(x)$: a real-valued function on $\mathbb{R}^d$ such $w(-x) = -w(x)$
- New density via modulation/perturbation of *baseline* $f_0$:

$$f(x) = 2\, f_0(x)\, G_0\{w(x)\} \qquad x \in \mathbb{R}^d$$

(Azzalini & Capitanio 2003 JRSS-B; Wang et al. 2004 Stat.Sin)

- Introduce a location parameter by a shift transformation
- Yields a simple tool for building many classes of distributions (typically parametric families, but allows semi-parametric, e. g. if $w$ is infinite dimensional odd polynomial)
- More general formulations are possible, some recalled later (instances of highly general construction by Jupp *et al.*, 2016 JMVA)

# Symmetry modulation, stochastic representation

- Stochastics representation:
  - let $Z_0 \sim f_0$ and $T \sim G_0$, independent variables,
  - let $Z = \begin{cases} Z_0 & \text{if } T \le G_0\{w(Z_0)\} \\ -Z_0 & \text{otherwise} \end{cases}$
  - then $Z \sim f$

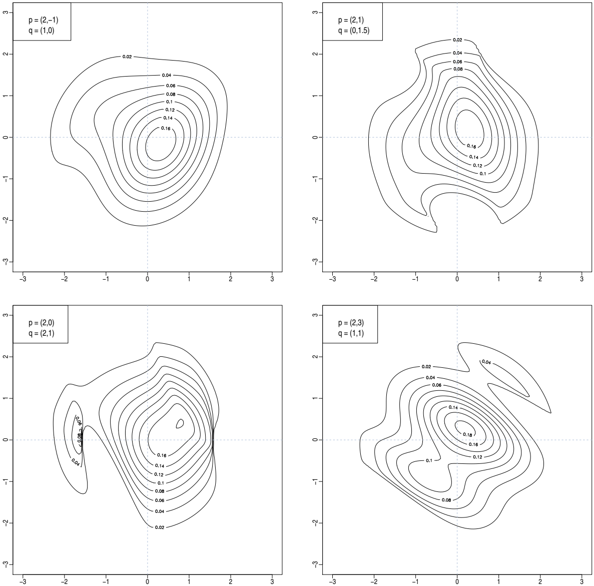
Additional forms of representation exist for specific instances

- This is useful for
  - deriving formal properties, such as 'perturbation invariance'

$$t(Z) \stackrel{d}{=} t(Z_0) \qquad \text{for any even function } t(\cdot)$$

  - random number generation
  - formulate models with subject matter motivation

# Symmetry modulation, examples with $f_0$ bivariate std normal



# Skew-elliptical distributions

# Recall elliptically contoured (EC) distributions

- Define 'elliptical' densities:

$$p_0(x) = \frac{c_d}{\det(\Sigma)^{1/2}}\,\tilde{p}\left((x-\mu)^\top \Sigma^{-1}(x-\mu)\right)$$

  where $c_d$ is a suitable normalizing constant, if integral exists

- Key fact: $p(x)$ is constant on ellipsoids, where

$$(x-\mu)^\top \Sigma^{-1}(x-\mu) = \text{constant}$$

- If $\mu = 0$, clearly $p_0(x) = p_0(-x)$

# Skew-elliptically contoured (SEC) distributions

- Combine the concepts of EC and symmetry-modulated distributions into

$$\begin{aligned} f(x) &= 2\, p_0(x)\; G_0\{w(x)\} \\ &= 2\, p_0(x)\; G(x) \qquad\qquad \text{say} \end{aligned}$$

- For any given $p_0$, many options for $G(x) = G_0\{w(x)\}$
- If $p_0$ is normal PDF $\varphi_d$, then 'the natural' choice is

$$2\,\varphi_d(x;\Sigma)\;\Phi(\eta^\top x), \qquad x, \eta \in \mathbb{R}^d$$

  yielding the skew-normal (SN) distribution
- The skew-normal is tractable superset of the normal family
- It preserves/extends *many* properties of the normal family
- For non-normal $p_0$, choice of $G(x)$ not so obvious

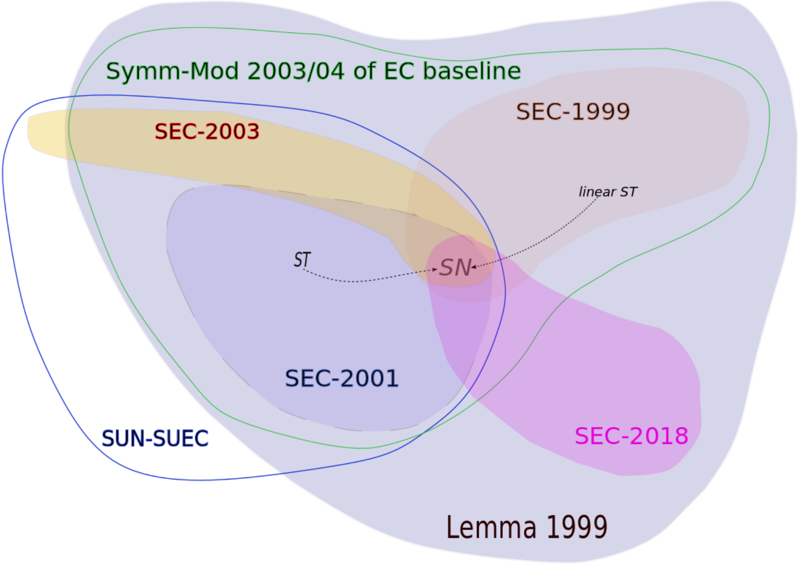
# Various SEC families, brief summary

- Lemma 1999 of Azzalini & Capitanio (1999, JRSS-B)
- SEC-1999: using this lemma, make a start with linear form

$$G(x) = G_0(\alpha\, x), \quad \text{for some appropriate } G_0$$

- Branco & Dey (2001) build on the conditioning mechanism – more later on
- use Symm-Mod 2003/04 construction with $f_0$ of EC type
- SEC-2003: Sahu, Dey & Branco use $d$-dimensional conditioning
- Arellano-Valle & Azzalini (2006) and Arellano-Valle & Genton (2010): SUEC construction emcompasses 2001-SEC, 2003-SEC and SUN too.
- additional constructions exists,
  e. g. Azzalini & Regoli (2018) going back to Lemma 1999

# Various SEC families, in a picture



# The 2001 SEC construction

- Start from the skew-normal distribution having density
$$2\,\varphi_d(x;\Sigma)\,\Phi(\eta^\top x) \qquad x,\eta \in \mathbb{R}^d$$
- A r.v. of this type can be represented as follows:
  let $X = (X_0, X_1, \ldots, X_d)$ normal with 0 mean and consider
$$(X_1, \ldots, X_d | X_0 > 0)$$
  which has a SN distribution
- Idea (Branco & Dey, 2001 JMVA):
  use this scheme whenever $X$ is elliptical, not only for normal
- Result (Azzalini & Regoli, 2012 AISM):
  the distribution so obtained is indeed a proper SEC,
  that is, a symmetry-modulated distribution of EC $p_0$

# The additive representation

- The SN distribution allows other stochastic representations
- For $U_0 \sim \mathrm{N}_d(0, \Psi)$ and $U_1 \sim \mathrm{N}(0, 1)$ independent r.v.'s

$$Z = D_\delta\, U_0 + \delta\, |U_1|, \qquad \delta \in \mathbb{R}^d,$$

  where $D_\delta = (\text{suitable diagonal matrix})$, has SN distribution
- This representation is of the form proposed by Simaan (1993, Management Sc.) to model non-symmetric security returns and to derive a number of theoretical results
- SEC distributions of the 2001 form also allow an additive representation as above, except that $U_0$ and $U_1$ are now uncorrelated. (Azzalini & Capitanio, 2003 JRSS-B)

# Features of the 2001 SEC family

- Allows two stochastic representations:
  - via conditioning
  - via additive construction
- Higher mathematical tractability,
  especially when baseline EC is a scale mixture of normals
- Other SEC's types do not achieve the same level of tractability

# The skew-$t$ (ST) distribution of 2001 SEC form

- A case of special interest is the 2001-SEC type skew-$t$ (ST):

$$(X_1, \dots, X_d | X_0 > 0) \quad \text{when } X \sim t_d(x; \nu)$$

- Workable expression of the density actually derived in 2003:

$$2\, t_d(x;\nu)\; T\{\mathrm{nonlinear}(x); \nu + 1\}, \qquad x \in \mathbb{R}^d$$

(independent papers of Azzalini & Capitanio and of AK Gupta)

- Nice formal properties of SEC-2001 hold and in addition:
  - additional stochastic representation as $\mathrm{SN}/\sqrt{\chi^2_\nu/\nu}$
  - explicit expression of moments up to order 4
  - family closed under marginalization and affine transformations
- Its 'extended' version, called EST, is obtained by

$$(X_1, \dots, X_d | X_0 > \tau) \quad \text{when } X \sim t_\nu$$

(2010, independent papers of Adcock and of Arellano-Valle & Genton)

- The EST distribution is also closed under conditioning

(at the cost of loosing perturbation invariance)

# SEC and related families in finance

# SN/ESN in finance and related areas

- SN and ESN families are mathematically very tractable
- one can extend normal-theory formulations allowing for skewness with limited extra complications
- Early such work in finance by Adcock & Shutes (2001) for portfolio selection under SN distribution of assets returns
- The additive representation $R = Y + \lambda|U|$ fits well within this logic, and it links to Simaan (1993) general formulation
- Much subsequent work along these lines:
  - more comprehensive follow-up work of Adcock (2004)
  - extension of Stein's lemma to ESN family (Adcock, 2007), introduced as a tool for optimization problems in finance
  - tail conditional expectation (Vernic, 2006)
  - model for asset pricing by Camichael & Coën (2013)
  - *et cetera. . .*
- In addition, representation by conditioning links precisely to Heckman selection model

# ST/EST/CEST in finance and related areas

- Since range of skewness of SN/ESN is limited, in certain cases ST/EST may be preferable
- CEST is a further extension, with $m$ hidden censoring variables
- Some features useful for flexible data fitting:
  - range of univariate skewness is $(-\infty, \infty)$
  - kurtosis in $[0, \infty)$ for ST, $[-c, \infty)$ for EST
  - infinite variance if $\nu \leq 2$
- Empirical explorations with real data confirm high flexibility
- Price is a diminished mathematical tractability
- Adcock (2010): asset pricing and portfolio selection for EST
- Adcock (2013): Stein's lemma for CEST and application to portfolio selection

# Discussion

# Pros and cons

- Although there are some differences, these constructions are closely related, sharing a common underlying logic
- Supersets of familiar parametric families,
  with additional regulating parameters
- The modulation mechanism retains some (sometimes many) properties of the original baseline distribution
- Overall effect is an increase of flexibility, while retaining interesting features and mathematical tractability
- The other side of the coin:
  there is a finite number of regulating parameters
  (unless we opt for an infinite-dimensional parameter — possible but hardly explored)

# Alternative parametric constructions

- There are very many, fewer allow multivariate form
- Even in the multivariate case only, a review here is impossible
- In finance, a popular tool is the two-piece construction
  - originated by Fechner (1897), . . . , Fernández & Steel (1988)
  - a simple and practical construction in the univariate case — key aspects of its popularity
  - but difficult to link to a ‘physical’ generating mechanism
  - hard to extend to the multivariate case

# Pros and cons of copulae

- Copulae separate modelling of dependence and marginals, achieving tremendous flexibility for data fitting
- The limitations of the resulting joint distributions are
  - lack of tractable properties, e.g. marginalization
  - no ‘physically motivated’ generating mechanism, hence no natural link with substantive theory
- Depending on the problem under consideration, these limitations may be relevant or not
- If mathematical tractability and/or link with a ‘physically motivated’ mechanism are important, parametric families like those presented here may be an attractive alternative.

# Resources

Monograph:

Azzalini, A. with the collaboration of A. Capitanio (2014). *The Skew-Normal and Related Families*, Cambridge University Press

Bibliography, software tools and other material available at
`http://azzalini.stat.unipd.it/SN`