Symmetry-modulated distributions: an introduction

Adelchi Azzalini
University of Padua, Italy

Université catholique de Louvain
26th January 2018

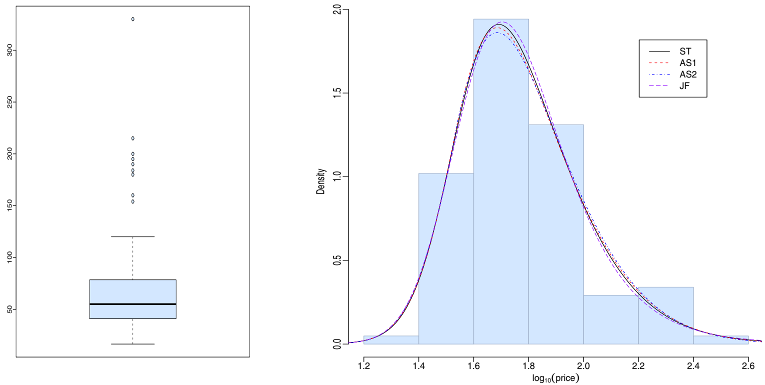
Context and aim

- A flood of probability distributions has surged in recent years
- Many proposals aim at 'generalizing' the normal family
- Features of interest: skewness and kurtosis
- Within this context, we present an introduction to a specific formulation: symmetry-modulated distributions
  (AKA skew-symmetric distributions)
- works equally in the univariate and the multivariate case
- focus is essentially on continuous distributions
  (discrete constructions are possible, but limited)

Some general considertions

- The success of the normal distribution originates by the combination of
  - 'physical-motivation' for its genesis
  - mathematical tractability
  - reasonable empirical adequacy in a range of situations
- We want to improve on flexibility, that is, empirical adequacy,
- . . . while retaining other appealing aspects as far as possible
- Alternative formulations may have different priorities

Which fitting distribution to choose?



- fitting log-price of a bottle of Barolo wine
- different formulations are numerically nearly equivalent
- which one to choose?
- numerical adequacy is not all that matters

Basic case: skew-normal distribution for $d = 1$

- a key feature of the normal density is symmetry . . . *always!*
- idea: perturb the $\mathrm{N}(0, 1)$ p.d.f. $\phi(x)$ by an adjustable factor:

$$f_{\mathrm{SN}}(x) = 2\,\phi(x)\,\Phi(\alpha\,x)$$

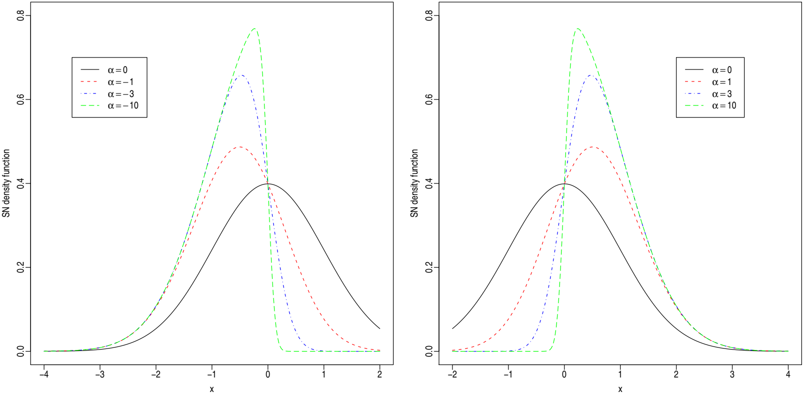
  where Φ is the $\mathrm{N}(0, 1)$ c.d.f. and α is a real parameter
- the normalizing factor 2 holds for all α's
- if $\alpha = 0$ reduce to $\mathrm{N}(0, 1)$
- in practical work introduce location and scale parameters:

$$Z \text{ has density } f_{\mathrm{SN}}, \qquad Y = \xi + \omega\,Z$$

  so that Y is regulated by ξ, ω, α.
- we say that Z, Y have skew-normal distribution, write

$$Z \sim \mathrm{SN}(0, 1, \alpha), \qquad Y \sim \mathrm{SN}(\xi, \omega^2, \alpha)$$

Skew-normal distribution – some examples



A number of nice properties

- explicit expression for moment generating function, hence moments
- manageable expression of the distribution function
- nice formal properties, a key instance is $Z^2 \sim \chi^2_1$
- various stochastic representations available
  - useful for random-number generation of SN variates
  - motivate the adoption of this model in specific situations
- the two more important stochastic representations are
  1. via a selection (censoring) mechanism:
     given $(X_0, X_1) \sim \mathrm{N}_2(0, P)$, take $Z = (X_0 | X_1 > 0)$
  2. via an additive form:
     given $U_0, U_1 \sim \mathrm{N}(0, 1)$ iid, take $Z = a|U_0| + b\, U_1$

Illustration: connection with 'stochastic frontier analysis'

- 'Stochastic frontier analysis' model for production units:

  $$(\text{product}) = f(\text{input factors}) - (\text{inefficiency}) + (\text{error term})$$

  (usually product is log-transformed)

- its basic version is of type

  $$(\text{product}) = f(\text{input factors}) \underbrace{-|\mathrm{N}_1| + \mathrm{N}_2}_{\text{random term}}$$

  for some independent 0-mean normal variables N_1 and N_2

- Recall additive representation, hence $-|\mathrm{N}_1| + \mathrm{N}_2 \sim \mathrm{SN}$
- this connection allows to make use of subsequent results to develop new tools for stochastic frontier analysis

Multivariate skew-normal distribution

- If $\phi_d(x;\bar\Omega)$ denotes $\mathrm{N}_d(0,\bar\Omega)$ p.d.f. where $\bar\Omega > 0$ has all 1's on the diagonal, then

$$2\,\phi_d(x;\bar\Omega)\,\Phi(x^\top\alpha), \qquad x\in\mathbb{R}^d,$$

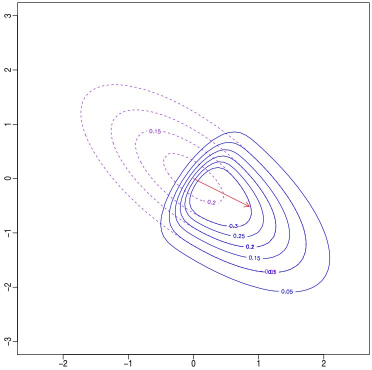
  is a density function for any vector parameter α
- if $\alpha = 0$ we are back to $\mathrm{N}_d(0,\bar\Omega)$, otherwise density is skew
- given Z distributed as above, consider the location-scale family generated by

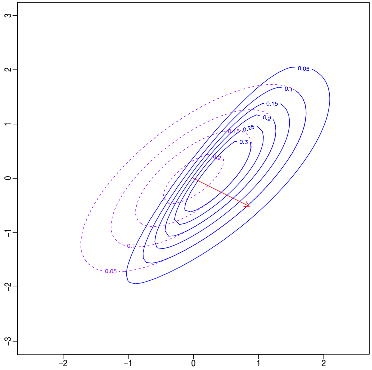
$$Y = \xi + \omega\,Z,$$

  where $\xi\in\mathbb{R}^d$ and ω is positive diagonal matrix
- we say that Y has a d-dimensional skew-normal distribution and write

$$Y \sim \mathrm{SN}_d(\xi,\Omega,\alpha), \qquad \Omega = \omega\bar\Omega\omega$$

Multivariate SN: some examples with $d = 2$





Multivariate SN: some properties and some uses

- family closed under marginalization and affine transformations:

  $a + A^\top Y \sim \mathrm{SN}_m(a + A^\top \xi, A^\top \Omega A, \tilde{\alpha}), \quad \tilde{\alpha} = \mathrm{function}(A, \bar{\Omega}, \alpha)$

- distribution of quadratic forms, e. g.

  $$\text{Mahalanobis distances} : (Y - \xi)^\top \Omega^{-1} (Y - \xi) \sim \chi^2_d$$

  a special case of a more general *exact* result on quadratic forms

- moment generating function of Z:

  $$M_Z(t) = 2 \exp(\tfrac{1}{2} t^\top \bar{\Omega} t)\, \Phi(\delta^\top t), \qquad \delta = \mathrm{function}(\bar{\Omega}, \alpha)$$

- this allows us to extend classical formulations for normal rv's

Formulations in finance and alike: from Normal to SN

- Adcock & Shutes (1999): CAPM under SN dist'n assuption
- Adcock (2007): extension of Stein's lemma
- Corns & Satchell (2007): skew Brownian motion, extend Black-Sholes formula for pricing options
- Carmichael & Coën (2013): asset pricing under SN returns
- De Luca & *et alii* (2004, 2005): multivariate GARCH-type model for asymmetric relationships among financial markets
- Vernic (2006): tail conditional expectation (for $d = 1$)
- *et cetera*

More general context: modulation of symmetry

- A more general form of modulation (or perturbation) of symmetry:

$$f(x) = 2\, f_0(x)\, G_0\{w(x)\}, \qquad x \in \mathbb{R}^d,$$

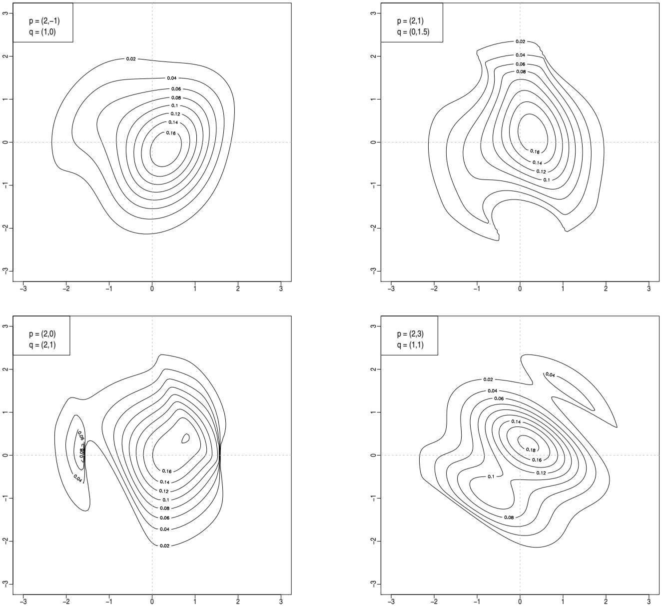
with conditions

  - f_0 is d-dimensional p.d.f., symmetric about 0: $f_0(x) = f_0(-x)$,
  - G_0 is symmetric continuous c.d.f on $\mathbb{R}$: $G_0(t) + G_0(-t) = 1$,
  - $w(x)$ is 'odd': $w(-x) = -w(x)$

always lends a proper density function on $\mathbb{R}^d$

- SN is a special case: $f_0(x) = \phi_d(x; \bar{\Omega})$, $G_0 = \Phi$, $w(x) = \alpha^\top x$
- The prescriptions are simple
$\rightarrow$ a wide (wild, perhaps) universe of constructions is possible

Wild perturbations of the standard bivariate Normal density



Skew-elliptical distributions

$$f(x) = 2\, f_0(x)\, G_0\{w(x)\}, \qquad x \in \mathbb{R}^d$$

- Theorem: all densities can be written in this form
- An interesing subclass: baseline f_0 is elliptical
- a further specification: scale mixtures of SN variates
- that is, $S\,X$ where $X \sim \mathrm{SN}_d(0, \Omega, \alpha)$ and $S > 0$ is indept r.v.
- an interesting case: $S \sim 1/\sqrt{\chi^2_\nu/\nu} \Rightarrow$ skew-t distribution (ST)

Skew-t distribution, I

- assume $Z \sim \mathrm{SN}_d(0, \bar\Omega, \alpha)$, $W_\nu \sim \chi^2_\nu$ indept; define skew-t r.v.:

$$\tilde Z = \frac{Z}{\sqrt{W_\nu/\nu}} \sim \mathrm{ST}_d(0, \bar\Omega, \alpha, \nu)$$

  similarly to construction of regular Student's t
- density of $\tilde Z$ is of type $2\, f_0(x)\, G_0\{w(x)\}$ where
  - f_0 is the multivariate t_ν density with 0 location
  - G_0 is the univariate $t_{\nu+d}$ c.d.f.
  - $w(x)$ is a suitable non-linear function of $(d, \bar\Omega, \alpha, \nu)$

  (a special instance of skew-elliptical family of distributions)
- limit behaviour as $\nu \to \infty$: $w(x) \to \alpha^\top x$ and $\mathrm{ST} \to \mathrm{SN}$
- include location and scale:

$$\tilde Y = \xi + \omega\, \tilde Z$$

- four-parameter skew-t distribution:

$$\tilde Y \sim \mathrm{ST}_d(\xi, \Omega, \alpha, \nu), \qquad \Omega = \omega \bar\Omega \omega$$

Skew-t distribution, II

- closure under marginalization and affine transformations holds
- Mahalanobis distances:

$$(Y-\xi)^\top\Omega^{-1}(Y-\xi) \sim \text{scaled } F$$

  useful to build model diagnostics
- no MGF, but moments computed via stochastic representation
  (only moments up to an order less than ν exist, like for usual Student's t)
- wide range of coefficients of skewness and kurtosis
- hence high flexibility to fit data
- in particular low ν's allow for long tails, possibly asymmetric

Skew-*t* distribution, III

- ST is a highly flexible distribution
- retains mathematical tractability, although reduced wrt SN
- applications in many research domains
- instances in econometrics/finance, empirical and theoretical:
  - Walls (2005) models (log-)returns of film industry
  - on similar theme, work of Pitt (2010, paper and monograph)
  - Meucci (2006) extends Black-Litterman technique
  - Adcock (2010) adapts his earlier work on portfolio selection
  - *et cetera*
- much use in finite mixtures/model-based clustering

- Beware of confusion: after **this skew-*t*** has been introduced in 2001, the name has been adoped for some different proposals.

Recap

- Overall target is to build flexible and tractable distributions
- SN and ST distributions are appealing in this logic
- software tools available
- if extra flexibility is required, general formulation offers the tool
  (at the cost of reduced tractability)

Resources and tools

- A. Azzalini with the collaboration of A. Capitanio (2014). *The Skew-Normal and Related Families*. Cambridge University Press, IMS Monographs series.
- Bibliography and other material at: `http://azzalini.stat.unipd.it/SN/`
- Software:
  - R package `sn` on CRAN
  - some other tools exist (e.g. in Matlab)