Selection models under generalized symmetry
(Return to Lemma 1)

Adelchi Azzalini

Università di Padova, Italia

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Skew-Symmetric Set (SSS) of distributions

Context: families of continuous distributions on $\mathbb{R}^d$

- 'base density' f_0 centrally symmetric around 0:

$$f_0(x) = f_0(-x), \qquad x \in \mathbb{R}^d$$

- an odd real-valued $w(x)$: $w(-x) = -w(x)$
- a scalar cdf $G(\cdot)$ with symmetric pdf $G'(\cdot)$
- *skew-symmetric* pdf:

$$f(x) = 2\, f_0(x)\, G\{w(x)\}$$

- equivalently

$$f(x) = 2\, f_0(x)\, \pi(x)$$

where $\pi(x) \geq 0$, $\pi(x) + \pi(-x) = 1$

Historical development

Note: symmetric = (centrally) symmetric about 0

- Lemma 1 (Azzalini & Capitanio, 1999):
  if $Y \sim f_0$, $w(Y)$ has symmetric pdf, G' symmetric pdf,
  $\Rightarrow 2\, f_0(x)\, G\{w(x)\}$ is a density
- Corollary (1999): if f_0 elliptical, $w(\cdot)$ linear, G' symmetric
  $\Rightarrow 2\, f_0(x)\, G\{w(x)\}$ is a skew-elliptical pdf
- extensions (2001$\longrightarrow$)
  weaker conditions: f_0 centrally symmetric, $w(\cdot)$ odd
  $\Rightarrow$ skew-symmetric distributions
- developments: lots, important and continuing

Re-consider Lemma 1

Lemma 1:
if $Y \sim f_0$, $w(Y)$ has symmetric pdf, G' symmetric pdf,
$\Rightarrow 2\, f_0(x)\, G\{w(x)\}$ is a density

- $f_0(\cdot)$ does not need to be symmetric
- $w(\cdot)$ does not need to be even
- $\Rightarrow$ get non(skew-symmetric) pdf
- ideally, search equivalent of representation

$$Z = \begin{cases} Y & \text{if } \; X \leq w(Y) \\ -Y & \text{otherwise} \end{cases}$$

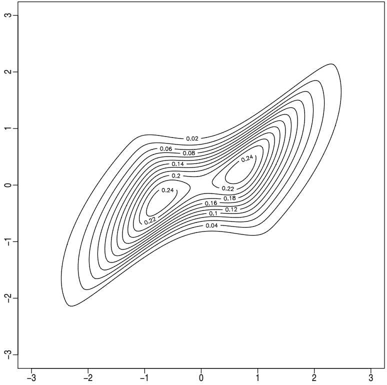
An example with $w(x)$ even

$$f(x) = 2\,\phi_2(x;\Omega)\,\Phi\left\{\alpha(x_1^2 - x_2^2)\right\}, \qquad x = (x_1, x_2) \in \mathbb{R}^2$$

- $w(x) = \alpha(x_1^2 - x_2^2)$ is even
- $\pi(x) = \Phi(w(x)) = \pi(-x)$
- hence $\pi(x) + \pi(-x) = 2\,\pi(x) \neq 1$
- but $w(Y)$ has symmetric pdf, if $Y \sim N_2(0, \Omega)$
- $\Rightarrow f(x)$ is a proper pdf

An example with $w(x)$ even, ctd

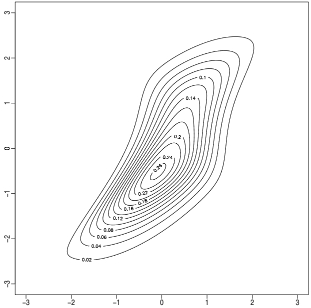
$$f(x) = 2\,\phi_2(x;\Omega)\,\Phi\left\{\alpha(x_1^2 - x_2^2)\right\}$$



Note it is non-skew: it is centrally symmetric!

An example with non-even non-odd $w(x)$

$$f(x) = 2\,\phi_2(x;\Omega)\,\Phi\{\alpha_1(x_1 - x_2) + \alpha_2(x_1^2 - x_2^2)\}$$



$$\alpha_1 = 1, \alpha_2 = -1, \rho = 2/3$$

Distribution of Arnold, Castillo & Sarabia (2002)

$$f(x) = 2\,\phi_2(x; I_2)\,\Phi(\alpha x_1\, x_2), \qquad x = (x_1, x_2) \in \mathbb{R}^2$$

- $w(x) = \alpha\, x_1\, x_2$ is even
- $w(Y)$ symmetric pdf, if $Y \sim N_2(0, I_2)$
- hence it belongs to this framework

An example with $f_0(x)$ non-symmetric

Need *some* symmetry, not necessarily $f_0(x) = f_0(-x)$

- Example: let consider Y_1, Y_2 independent positive rv's,

$$Y_j \sim h(y), \qquad y \in \mathbb{R}^+$$

  For instance $Y_j \sim$Gamma, non-symmetric

- $w(Y) = \alpha(Y_1 - Y_2)$ has symmetric density
- $\Rightarrow$

$$f(x) = 2 \underbrace{h(x_1, \omega)\ h(x_2; \omega)}_{f_0(x)}\ G\{\alpha(x_1 - x_2)\}, \quad x = (x_1, x_2) \in \mathbb{R}^+ \times \mathbb{R}^+$$

  is a proper pdf for any G' symmetric

A general scheme (*not* 'The')

- Assume there exists an invertible transformation $R(\cdot)$ such that

$$\underbrace{f_0(y) = f_0[R(y)]}_{\text{generalized symmetry}}, \quad |\det R'(y)| = 1, \quad w[R(y)] = -w(y)$$

- If $Y \sim f_0$ and $X \sim G'$, independent, then

$$Z = \begin{cases} Y & \text{if } X \leq w(Y) \\ R^{-1}(Y) & \text{otherwise} \end{cases}$$

  has pdf $f(z) = 2\, f_0(z)\, G[w(z)]$

- Distributional invariance property holds:

$$t(Y) \stackrel{d}{=} t(Z)$$

On the trasformation $R(\cdot)$

- Special case:
  if $R(y) = -y$, get 'classical' skew-symmetric pdf
- If R is one suitable transformation, R^{-1} is another one
- Sometimes $R = R^{-1}$, for instance

$$R(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} y, \qquad y \in \mathbb{R}^2$$

- Often rotation transformations play the role
- Question: how to find suitable R's?

Conclusions

- Lemma 1 encapsulates more distributions than the skew-symmetric set (SSS)
- The complement subset $S_{\mathrm{Lemma\,1}} \backslash \mathrm{SSS}$ includes interesting cases
- Some of them have been shown here, but many others must exist

References

- Azzalini & Capitanio (1999). J.Roy.Stat.Soc. B, **61**, 579–602.
- Azzalini (2009). `http://arXiv.org:0912.5303`