# On MLE boundary values for skew-symmetric distributions

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# Skew-normal distribution and friends

- Basic form: skew-normal (SN) random variable $Z$ with pdf

$$f(z) = 2\,\phi(z)\,\Phi(\alpha\, z), \qquad z \in \mathbb{R}$$

- friends of the form

$$f(z) = 2\,f_0(z)\,G_0\{w(z)\}$$

  where

$$f_0(x) = f_0(-x), \quad G_0'(x) = G_0'(-x), \quad w(-x) = -w(x)$$

- add location and scale parameter

$$Y = \xi + \omega\, Z$$

- multivariate versions exist

# Two sides of the coin

Two sides of the coin:

- formulation allows nice treatment of probability side
- statistical side somewhat peculiar aspects

Challenging side:

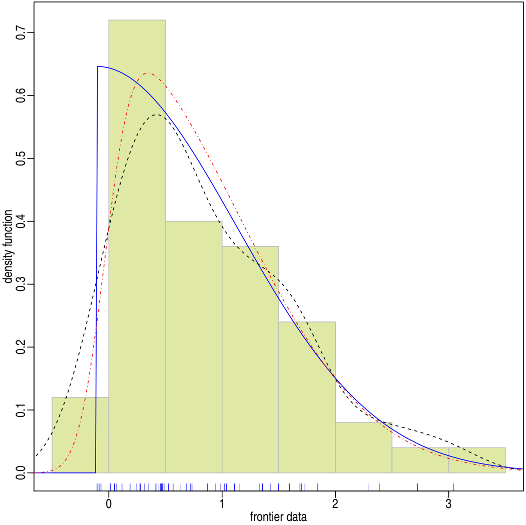
1. under SN model, Info$(\xi, \omega, \alpha)$ is singular at $\alpha = 0$
2. for finite samples $\mathbb{P}\{\hat{\alpha} = \pm\infty\} > 0$

Deal with problem No. 2

# One parameter case, $d = 1$

- $$\begin{aligned} f(z) &= 2\,\phi(z)\,\Phi(\alpha z) \\ \log L(\alpha) &= \text{const} + \textstyle\sum_{i=1}^{n} \log \Phi(\alpha z_i) \end{aligned}$$
- $\log L$ monotone if all elements are of equal sign
- (monotone but bounded!)
- $$\begin{aligned} p_{n,\alpha} &= \mathbb{P}\{\hat{\alpha} = \pm\infty\} \\ &= \left(\frac{1}{2} - \frac{\arctan\alpha}{\pi}\right)^n + \left(\frac{1}{2} + \frac{\arctan\alpha}{\pi}\right)^n \end{aligned}$$
- e.g. $p_{25,5} \approx 0.197$ and $p_{50,5} \approx 0.039$.

# A three-parameter example: frontier data




$n = 50$ values from $\mathrm{SN}(0, 1, 5)$, fit $\mathrm{SN}(\xi, \omega^2, \alpha)$

# Options and remarks

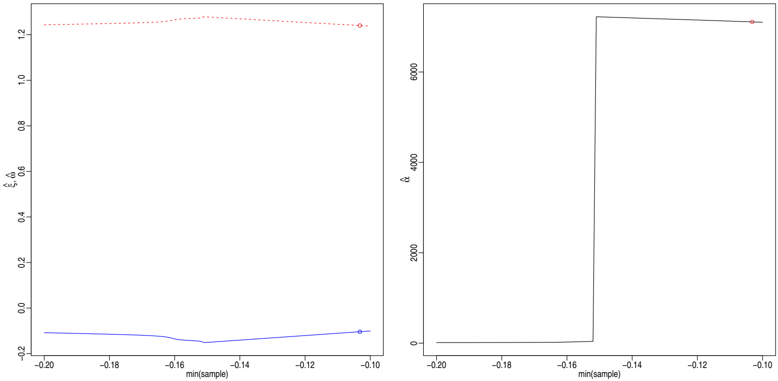
Alternative routes:

1. live with MLE as it is (must learn how!)
2. look for alternatives/adjustments

Remarks on illustrative ‘frontier’ data:

- histogram & nonparameteric $\hat{f}$ not like half-hormal
- $\hat{\gamma}_1 = 0.902$ inside admissible region $(-0.995, 0.995)$
- MLE behaves discontinuously

# Frontier data, MLE's vs min(sample)



# Sartori-Firth method of bias reduction

- Firth (1993) method for bias reduction: solve

$$\ell'(\alpha) + M(\alpha) = 0$$

where

$$M(\alpha) = -i(\alpha)\, b(\alpha)$$

- in our case bias is infinite!
- Sartori (2006):

$$M(\alpha) = -\frac{\alpha}{2}\,\frac{a_4(\alpha)}{a_2(\alpha)}$$

$$a_p(\alpha) = \mathbb{E}\left\{Z^p\,\zeta_1(\alpha\,Z)^2\right\}, \qquad \zeta_1(x) = \frac{\phi(x)}{\Phi(x)}$$

- needs two numerical integrations for each function evaluation
- extension to three-parameter case not easy

# Bayesian approach

- prior $\pi(\alpha)$ avoids MAP at $\alpha = \pm\infty$
- Jeffreys' prior $\pi_J(\alpha)$ is a proper distribution
- in three-parameter, expression of reference-integrated likelihood is known, but not usable in practice
- a proposed approximation

$$\pi_J(\alpha) \approx \text{const} \times \left(1 + \frac{2\alpha^2}{\pi^2/4}\right)^{-3/4}$$

  a scaled $t(1/2)$ distribution
- this is numerically close to $M(\alpha)$
- in practice inference via Gibbs sampling

References: Liseo & Loperfido (2006), Bayes & Branco (2007)

# Penalized log-likelihood and MPLE

- Consider penalized log-likelihood

$$\ell_p(\theta) = \log L_p(\theta) = \log L(\theta) - Q(\theta)$$

where $\theta$ is the parameter set with 1 or 3 (or more) components

- penalty $Q$ such that:

$$Q \geq 0, \quad Q\big|_{\alpha=0} = 0\,, \qquad \lim_{\alpha \to \pm\infty} Q = \infty$$

$$Q = \mathcal{O}_p(1) \qquad \text{as } n \to \infty$$

- recall that $\log L$ is bounded

$\Longrightarrow \qquad \tilde{\theta} = \arg\max_\theta \ell_p(\theta)$ exists

$\tilde{\theta} \equiv$ MPLE

# Basic asymptotics

- $\hat{\theta}$ is MLE, $\tilde{\theta}$ is MPLE
- $$\begin{aligned} \tilde{\theta} - \hat{\theta} &= \ell_p''(\hat{\theta})^{-1} Q'(\hat{\theta}) + \text{remainder} \\ &= \mathcal{O}_p(n^{-1}) \end{aligned}$$
- $$\mathrm{var}\left\{\tilde{\theta}\right\} \approx -\ell_p''(\tilde{\theta})^{-1}$$

# Choosing $Q$

- 'natural' proposal for $Q$

$$Q = c_1 \log(1 + c_2\, \alpha^2), \qquad c_1, c_2 > 0$$

- equate

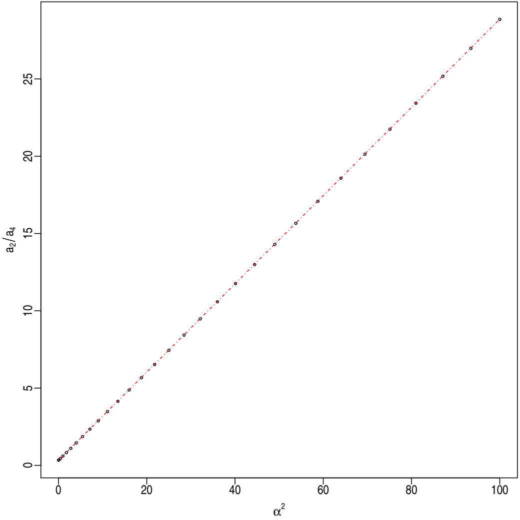
$$\ell_p'(\theta) = \ell'(\theta) - Q'(\alpha) = \ell'(\theta) + M(\alpha)$$

- write

$$-\frac{\alpha}{2\, M(\alpha)} = \frac{a_2(\alpha)}{a_4(\alpha)} \approx e_1 + e_2\, \alpha^2$$

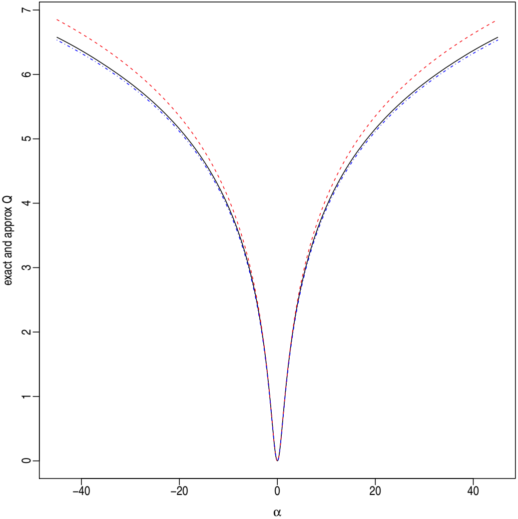
- find $e_1$ and $e_2$ by matching limits at $\alpha^2 = 0$ and $\alpha^2 \to \infty$
- $c_1 = 1/(4\, e_2), \quad c_2 = e_2/e_1$

# Linearization of $a_2/a_4$



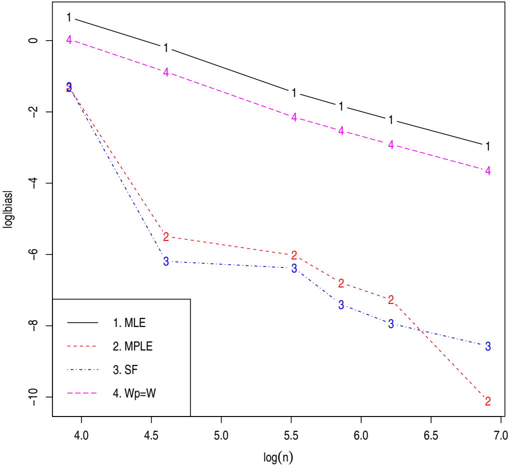

dots: exact points, red line: linearized expression

# Penalty $Q$ and approximations




exact $Q$, Bayes-Branco approx., using linearization

# Simulation work




estimate $\mathrm{SN}(\xi, \omega^2, \alpha)$ when sample is from $\mathrm{SN}(0, 1, 5)$

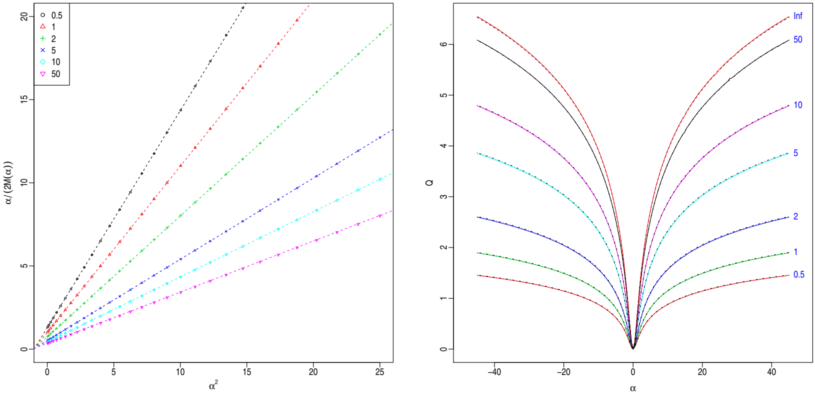
# Skew-$t$ distribution

- pdf:

  $$f(x) = 2\,\omega^{-1}\, t(z;\nu)\, T\{w(z);\nu+1\}, \qquad z = \omega^{-1}(x-\xi) \in \mathbb{R}$$

- proceed as for SN case, but $\nu$ affects coefficients
-
  $$-\frac{\alpha}{2\,M(\alpha)} \approx e_{1\nu} + e_{2\nu}\,\alpha^2$$

# Penalty $Q$ and approximations for ST



# SN distribution in $d$-dimensions

- pdf:

$$f(x) = 2\,\phi_d(x-\xi;\Omega)\,\Phi(\alpha^\top \omega^{-1}(x-\xi)), \qquad x \in \mathbb{R}^d$$

- many aspects encapsulated in summary quantity

$$\alpha_* = \left(\alpha^\top \bar\Omega \alpha\right)^{1/2}, \qquad \text{where } \bar\Omega = \omega^{-1}\Omega\omega^{-1}$$

- use penalty

$$Q = c_1 \log(1 + c_2\,\alpha_*^2)$$

- do similarly for the multivariate skew-$t$ distribution

# Final comments

- penalized log $L$ is linked to earlier work for specific cases
- in basic cases, MPLE essentially coincident with SF
- but MPLE is of more general applicability, within this context (possibly outside)
- MPLE can be combined with parameter transformations

# References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