Stochastic representations and other properties of skew-symmetric distributions

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Overview

- a brief introduction to skew-symmetric distributions
- stochastic representations and some implications
- a few important subclasses
- other interesting properties

Skew-symmetric distributions: basic construct

Start from a symmetric 'base' PDF, and modify it

If

f_0 *a (centrally) symmetric density in* $\mathbb{R}^d$*:* $f_0(x) = f_0(-x)$

G_0 *univariate CDF, such that* G' *is even*

w *odd real-valued function:* $w(-x) = -w(x)$

then

$$f(x) = 2\, f_0(x)\, G_0\{w(x)\}$$

is a proper density function on $\mathbb{R}^d$*.*

- The proof is surprisingly simple (next slide)
- In practical work include location and scale parameters, dependence on covariates, etc
- Several variants and further extensions exist.

Proof

$$f(x) = 2\, f_0(x)\, G_0\{w(x)\}$$

The proof that $\int f = 1$ is elementary *and* instructive.

- take $Y \sim f_0$ and $W = w(Y)$.
- for any Borel set A with 'mirror set' $-A$,

  $$\mathbb{P}\{W \in -A\} = \mathbb{P}\{-W \in A\} = \mathbb{P}\{w(-Y) \in A\} = \mathbb{P}\{w(Y) \in A\}$$

  $$\implies \quad W \text{ symmetric about } 0$$
- If $X \sim G_0'$, independent of Y, then $X - W$ symmetric about 0.
- Hence

  $$\tfrac{1}{2} = \mathbb{P}\{X \le W\} = \mathbb{E}_Y\{\mathbb{P}\{X \le w(Y) | Y\}\} = \int_{\mathbb{R}^d} G_0(w(y))\, f_0(y)\, \mathrm{d}y$$

QED

An essentially equivalent formulation

$$f(x) = 2\, f_0(x)\, G(x), \qquad \text{where } G(x) = G_0\{w(x)\}$$

- properties: $G(x) \geq 0$ and $G(x) + G(-x) = 1$, for any $x \in \mathbb{R}^d$.
- Can do the development using $G(x)$ with above properties.
- Essentially equivalent constructions.
- Work with G is mathematially neater
- Actual specification of G is convenient via $G(x) = G_0\{w(x)\}$

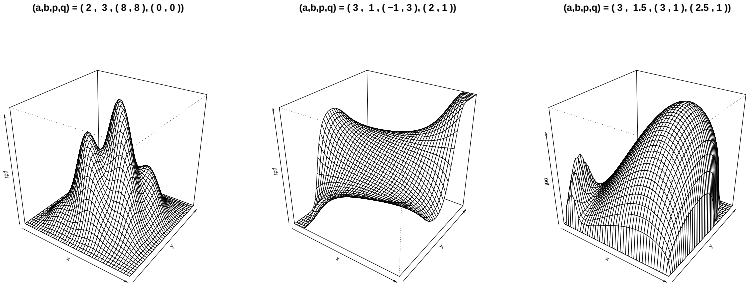
An example: perturbed Beta distribution

$$f(x) = 2\, f_0(x)\, G_0\{w(x)\}, \qquad x \in (-1,1)^2$$

- f_0 the product of two symmetric Beta densities in $(-1,1)$, with parameters a and b
- G_0 the standard logistic distribution function
- w let

$$w(y) = \frac{\sin(p_1 y_1 + p2 y_2)}{1 + \cos(q_1 y_1 + q_2 y_2)}$$

Example (ctd): perturbed Beta distribution



Important case: The skew-normal distribution (SN)

$$f(x) = 2\,\phi_d(x;\bar\Omega)\,\Phi(\alpha^\top x), \qquad (x \in \mathbb{R}^d)$$

where

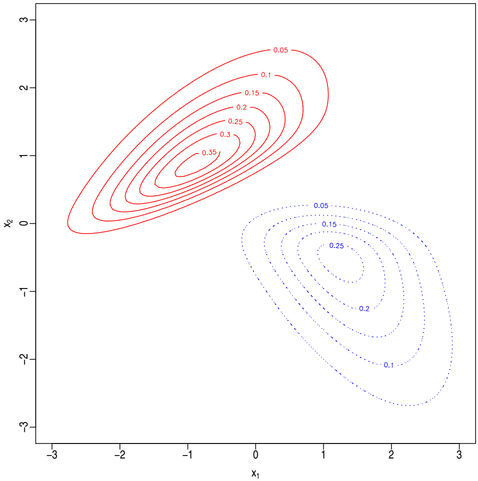
- $\phi_d(x;\bar\Omega)$ denotes $\mathrm{N}_d(0,\bar\Omega)$ pdf, $\quad\bar\Omega$ a correlation matrix
- $\Phi(\cdot)$ denotes $\mathrm{N}(0,1)$ cdf
- $\alpha \in \mathbb{R}^d$ is a vector of shape parameters

Allow for location and scale: if $Z \sim f(\cdot)$, let

$$Y = \xi + \omega\,Z, \qquad (\omega = \text{diagonal matrix} > 0)$$

Parameters: ξ (location), $\quad\Omega = \omega\bar\Omega\omega$ (scale), $\quad\alpha$ (shape)

SN distribution (ctd)



A general stochastic representation

$$f(x) = 2\, f_0(x)\, G_0\{w(x)\}$$

want stochastic representation for $Z \sim f$

- The proof that $\int f = 1$ implies that
$$Z = Y | X < w(Y)$$
if $X \sim G'$, $Y \sim f_0$, independent.
- When $X \geq w(Y)$, then $-X \leq w(-Y)$, and $-Y \sim f$.
- Hence
$$Z = \begin{cases} Y & \text{if } X \leq w(Y) \\ -Y & \text{otherwise} \end{cases}$$
- Alternative proof by direct calculation

Perturbation invariance property

$$\begin{aligned} Y &\sim f_0(x) \\ Z &\sim 2\, f_0(x)\, G_0(w(x)) \end{aligned}$$

- Recall: $Z = \begin{cases} Y & \text{if } X \le w(Y) \\ -Y & \text{otherwise} \end{cases}$
- Corollary: $\boxed{t(Z) \stackrel{d}{=} t(Y) \text{ for any even function } t(\cdot)}$
- Lots of implications, including:
  - equality of even order moments,
  - the same distribution of quadratic forms
  - if t is q-dimensional, independence is preserved

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Skew elliptical distributions

- elliptical distribution (with 0 mean):

$$f_0(x) = \frac{c_d}{|\Omega|^{1/2}}\, \tilde{f}(x^\top \Omega^{-1} x)$$

- broad-sense 'skew-elliptical':

$$f(x) = 2\, f_0(x) G_0\{w(x)\}, \qquad \text{where } f_0 \text{ is elliptical}$$

- narrow-sense 'skew-elliptical': start from $(1+d)$ dimensional elliptical variate (Y_0, Y) and take

$$Z = \begin{cases} Y & \text{if } Y_0 > 0 \\ -Y & \text{if } Y_0 < 0 \end{cases}$$

- narrow-sense 'skew-elliptical' $\subset$ broad-sense 'skew-elliptical'

Skew elliptical distributions (ctd.)

- (narrow-sense) skew-elliptical variate are build with an additional stochastic representation

$$Z = \begin{cases} Y & \text{if } Y_0 > 0 \\ -Y & \text{if } Y_0 < 0 \end{cases}$$

- And another one:

$$Z_j = \delta_j |Y_0| + \sqrt{1-\delta_j^2}\, Y_j, \qquad j = 1, \ldots, d$$

  for suitable δ_j's depending on the distribution of (Y_0, Y)

- For $d = 2$, yet another one

$$Z = \max(Y_0, Y)$$

A wealth of other properties exist

- Any density function f admits a skew-symmetric representation:

$$f_0(x) = \frac{f(x) + f(-x)}{2}, \qquad G(x) = \frac{f(x)}{2\, f_0(x)}$$

- Perturbation invariance property is a characterization: i.e. $t(X) \overset{d}{=} t(Y)$ for all t, then X and Y share the same base f_0 in their skew-symmetric representation
- in the case $d = 1$, a stochastic ordering of distributions with common f_0 and varying G can be established, for suitable G's
- . . . lots more

Recap

- a vast family: skew-symmetric distributions (SSD)
- the aim is to achieve flexibility, not 'skewness'
- SSD allow a stochastic representation with usefull implications
- an important subset: skew-elliptical distributions (SED)
- SED family has a stronger structure
  and it allows additional stochastic representations
- further appealing properties hold, especially for SED

References

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- Wang, Boyer & Genton (2004, Stat. Sinica)
- Azzalini (2005, SJS), a review paper
- Azzalini & Regoli (2011, to appear in AISM)